

# Vacuum Nodes and Anomalies in Quantum Theories \*

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## Abstract

We show that nodal points of ground states of some quantum systems with magnetic interactions can be identified in simple geometric terms. We analyse in detail two different archetypical systems: i) the planar rotor with a non-trivial magnetic flux  $\Phi$  and ii) the Hall effect on a torus. In the case of the planar rotor we show that the level repulsion generated by any reflection invariant potential  $V$  is encoded in the nodal structure of the unique vacuum for  $\theta = \pi$ . In the second case we prove that the nodes of the first Landau level for unit magnetic charge appear at the crossing of the two non-contractible circles  $\alpha_-$ ,  $\beta_-$  with holonomies  $h_{\alpha_-}(A) = h_{\beta_-}(A) = -1$  for any reflection invariant potential  $V$ . This property illustrates the geometric origin of the quantum translation anomaly.

## 1 Introduction

Classical configurations play different roles in the description of quantum effects. Monopoles, skyrmions, vortices, solitons, kinks and similar classical configurations contribute to unveil the existence of non-trivial sectors in the energy spectrum of many quantum theories. Classical configurations are also important for the description of superselection sectors and non-trivial phase structures in quantum field theories. Tunnel effect is semiclassically described by means of instantons. There is, however, another kind of classical configurations which play a genuine quantum role in the description of some physical effects: the nodal configurations of physical states. It is known that the structure of those nodes encode information about relevant physical properties such as the complete integrability or chaotic behaviour of the corresponding systems.

Standard minimum principle arguments disfavor the appearance of nodes in ground states of quantum systems. However, the presence of CP violating interactions invalidates the use of such arguments and the vacuum response to this kind of interactions may involve the appearance of nodes. The existence of a non-trivial nodal structure in the vacuum states of quantum theories with CP violating interactions provides a new perspective in the analysis of the role of classical configurations in the quantum theory. In some cases, the infrared behaviour of the theory is so dramatically modified by the CP violating interaction that a confining vacuum state can become non-confining. The connection between the absence of confinement and the existence of nodes in the vacuum state, suggests that new classical field configurations related to the nodal structure of the quantum vacuum emerge as new

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candidates to play a significant role in the mechanism of confinement. This idea has been successfully exploited to show the absence of spontaneous breaking of CP symmetry at  $\theta = \pi$  for various field theories [6] [7].

In this paper we analyse the connection between the appearance of nodes in ground states of quantum systems generated by CP violating interactions and some non-perturbative quantum effects. In particular, we analyse in some detail the vacuum nodal structure of the quantum planar rotor with a  $\theta = \pi$  term and the quantum Hall effect on a torus. In both cases the vacuum nodal structure turns to be intimately related to the behaviour of the corresponding ground states under CP symmetry or translation symmetry.

In order to understand the physical origin of vacuum nodes let us briefly recall the standard argument which prevents the vanishing of the vacuum.

Let us consider a quantum system evolving on a finite dimensional Riemannian manifold  $(M, g)$  with Hamiltonian

$$H = \frac{1}{2}\Delta + V(x), \quad (1.1)$$

defined by the Laplace-Beltrami operator  $\Delta$  and a non-singular potential  $V(x)$ . Unitarity of quantum evolution requires the potential  $V(x)$  to be real,  $V(x) = V(x)^*$ , to guarantee the hermicity of  $H$ . In this case the system is invariant under time reversal,

$$U(T)\psi(x, t) = \psi(x, -t)^*,$$

because  $[U(T), H] = 0$ . This symmetry implies that for any energy level there is a basis of real stationary states,  $\psi_n(x) = \psi_n(x)^*$ . Indeed, if  $\psi_n(x)$  is an eigenstate of  $H$  with energy  $E_n$ , the state  $\psi_n(x)^*$  is also an eigenstate with the same energy. If  $\psi_n$  is not real,  $\psi_n^* \neq \psi_n$ , the states  $\psi_{\pm} = \psi_n^* \pm \psi_n$  will have the same energy  $E_n$  and will be real, irrespectively of the degeneracy or not of the energy level.

If  $H$  is bounded below and has a non-trivial discrete spectrum there is a ground state  $\psi_0$  whose energy attains the minimum  $E_0$  of the energy spectrum. Because  $V$  has no singularities on  $M$  it is trivial to see that  $\psi_0$  cannot vanish for any point  $x$  of the configuration space. Indeed,  $\psi_0$  satisfies the stationary equation

$$H\psi_0 = E_0\psi_0.$$

If the set of nodal points of  $\psi_0$ ,  $\mathcal{N}_0 = \{x \in M; \psi_0(x) = 0\}$ , is non-trivial, the positive real function  $\psi_1 = |\psi_0|$  is smooth everywhere except at the points of  $\mathcal{N}_0$ . The expectation value of the Hamiltonian on the state  $\psi_1$  is again  $E_0$  because the delta function singularity of  $\Delta\psi_1$  at  $\mathcal{N}_0$  is cancelled by the vanishing of  $\psi_1$  at that point,

$$E_0 = \langle \psi_0 | H \psi_0 \rangle = \langle \psi_1 | H \psi_1 \rangle;$$

e.g. in one dimension, if there is a nodal point,  $\mathcal{N}_0 = \{x_*\}$ , we have

$$\langle \psi_0 | \frac{d^2}{dx^2} \psi_0 \rangle = \langle \psi_1 | \frac{d^2}{dx^2} \psi_1 \rangle - 2 \int_{-\infty}^{\infty} \psi_0(x)^* \delta(x - x_*) \frac{d}{dx} \psi_0(x) = \langle \psi_1 | \frac{d^2}{dx^2} \psi_1 \rangle .$$

Since  $E_0$  is the lowest eigenvalue of  $H$  this means that  $\psi_1$  is also an eigenstate of  $H$  with the same energy that  $\psi_0$ . Now, elliptic regularity implies that any eigenstate of  $H$  must be smooth, thus  $\psi_1$  cannot be an eigenstate because its differential is discontinuous at  $\mathcal{N}_0$ . The contradiction, being motivated by the assumption of existence of nodes, disappears if  $\psi(x) \neq 0$  for any  $x \in M$ . The same argument leads to the proof of vacuum uniqueness. If the vacuum were degenerate, there would exist another ground state  $\psi_1 \neq \psi_0$ . Then, the ground state defined by  $\chi(x) = \psi_0(x_*)\psi_1(x) - \psi_1(x_*)\psi_0(x)$  will vanish for  $x = x_*$ , which cannot occur by the previous argument.

Both results rely heavily on the real, local and smooth characteristics of the potential  $V$ . Exceptions for this archetypical infrared behaviour of quantum systems can arise either by the introduction of internal degrees of freedom (e.g. spin), singular or non-local potentials, or complex interactions.

Complex interactions are physically generated by the presence of magnetic fields. The interaction with the magnetic gauge field potential  $A$  through the gauge principle of minimal coupling leads to a Hamiltonian

$$H_A = \frac{1}{2}\Delta_A + V(x), \quad (1.2)$$

which is not invariant under time reversal,  $U(T)H_AU(T) = H_{-A}$ . The eigenstates are not necessarily real functions and the rest of the argument leading to the absence of nodes and uniqueness of the vacuum state fails.

## 2 The Planar Rotor

Let us consider the case of a charged particle moving on a circle under the action of a periodic potential  $V(\varphi)$  and a non-trivial magnetic flux  $\Phi = 2\pi\epsilon$  crossing through the circle. In this case,  $M = S^1$  and

$$H_\epsilon = -\frac{1}{2}(\partial_\varphi - i\epsilon)^2 + V(\varphi), \quad (2.1)$$

where  $\varphi \in [-\pi, \pi)$  is the angular coordinate of the circle, and we assume that the mass and charge of the particle are  $m = e = 1$ .

### PROPOSITION 2.1

If the potential  $V$  is reflection invariant  $V(\varphi) = V(-\varphi)$ , the matrix element

$$K_T^\epsilon(\varphi_0, \varphi_1) = \langle \varphi_0 | e^{-TH_\epsilon} | \varphi_1 \rangle$$

of the heat kernel operator vanishes for  $\epsilon = 1/2$  when  $\varphi_0 = 0$  and  $\varphi_1 = \pi$ , i.e.

$$K_T^{1/2}(0, \pi) = 0.$$

*Proof.* In such a case the Hamiltonian (2.1) is invariant under the Bragg reflection symmetry

$$U(P)\psi(\varphi) = \psi(-\varphi)$$

and it is always possible to find in the Hilbert space  $\mathcal{H} = L^2(S^1)$  a complete basis of stationary states with definite  $U(P)$  symmetry. If the energy level is not degenerate, the corresponding physical state  $\psi(\varphi)$  has to be either even or odd under  $U(P)$  symmetry. In the degenerate case, if  $U(P)\psi$  is not the same state that  $\psi$ , the stationary functionals  $\psi_{\pm} = \psi \pm U(P)\psi$  are parity even/odd, respectively. This implies that kernel element  $K_T^{1/2}(0, \pi)$  is reflection invariant

$$U(P)^\dagger K_T^{1/2} U(P)(0, \pi) = \sum_n U(P)\psi_n(0)^* U(P)\psi_n(\pi) e^{-E_n T} = K_T^{1/2}(0, \pi), \quad (2.2)$$

On the other hand in the path integral representation

$$K_T^\epsilon(\varphi_0, \varphi_1) = \int_{\substack{\varphi(0)=\varphi_0 \\ \varphi(T)=\varphi_1}} \delta\varphi \exp \left\{ - \int_0^T dt \left[ \frac{1}{2} \dot{\varphi}(t)^2 + i\epsilon \dot{\varphi}(t) + V(\varphi(t)) \right] \right\}. \quad (2.3)$$

we have that

$$U(P)^\dagger K_T^\epsilon U(P)(0, \pi) = K_T^{-\epsilon}(0, \pi) \quad (2.4)$$

because the  $P$  transformation leaves the points  $\varphi = 0$  and  $\varphi = \pi$  invariant but changes the sign of the  $\epsilon$ -term in the exponent of the path integral, since it reverses the orientation of every path. The contribution of this term becomes  $-2\pi i\epsilon(1/2 + n)$  instead of  $2\pi i\epsilon(1/2 + n)$  for any trajectory  $\varphi(t)$  in  $S^1$  connecting  $\varphi = 0$  with  $\varphi = \pi$  with winding number  $n$ . Thus, the kernel element  $K_T^\epsilon(0, \pi)$  is not invariant under reflection symmetry, unless  $\epsilon = 0 \pmod{\mathbb{Z}}$ . In particular for  $\epsilon = 1/2$ , the kernel element  $K_T^{1/2}(0, \pi)$  is parity odd and purely imaginary

$$U(P)^\dagger K_T^{1/2} U(P)(0, \pi) = K_T^{-1/2}(0, \pi) = K_T^{1/2}(0, \pi)^* = -K_T^{1/2}(0, \pi).$$

This is in disagreement with (2.2) unless the kernel element vanishes for those points  $K_T^{1/2}(0, \pi) = 0$ .

This property is independent of the potential term  $V$  and the value of  $T$ . In particular, it implies that the same vanishing property holds for the restriction of  $K_T(0, \pi)$  to any energy level, e.g the ground state. If the vacuum is non degenerate it has to vanish either at  $\varphi = \pi$  or  $\varphi = 0$  for this particular value  $\Phi = \pi$  of the magnetic flux ( $\epsilon = 1/2$ ).

This property of the heat kernel can also be understood in the Hamiltonian formalism. The presence of the magnetic flux has a non-trivial effect in the energy spectrum of the theory (Aharonov-Bohm effect) because of the non-simply connected character of  $S^1$ ,  $\pi_1(S^1) = \mathbb{Z}$ . Although the  $\epsilon$  dependence cannot be removed by a globally defined gauge transformation, the singular gauge transformation

$$\xi(\varphi) = e^{-i\epsilon\varphi} \psi(\varphi) \quad (2.5)$$

which is uniquely defined on the domain  $(-\pi, \pi)$  but is discontinuous at  $\varphi = \pm\pi$ , removes the  $\epsilon$  dependence of quantum Hamiltonian

$$\tilde{H}_\epsilon = e^{-i\epsilon\varphi} H_\epsilon e^{i\epsilon\varphi} = H_0.$$

The  $\epsilon$  dependence is, however, encoded in the non-trivial boundary conditions that physical states have to verify at the boundary  $\varphi_{\pm} = \pm\pi$ ,

$$\xi(-\pi) = e^{-i\epsilon}\xi(\pi). \quad (2.6)$$

In this sense the transformation is trading the  $\epsilon$ -dependence of the Hamiltonian for non-trivial boundary conditions at  $\varphi_{\pm}$ .

The relevant extra property which allows us to extract some information on the nodal structure of the quantum vacua is that the theory is  $U(P)$  invariant for  $\epsilon = 1/2$ . In the Hamiltonian approach this property of the special case  $\epsilon = 1/2$  comes from the fact that the boundary condition (2.6), becomes an anti-periodic boundary condition,  $\xi(\varphi_+) = -\xi(\varphi_-)$ , which is a reflection invariant condition.

As discussed above it is always possible to find a complete basis of stationary states with definite  $U(P)$  symmetry. If the energy level is not degenerate the corresponding physical state  $\psi(\varphi)$  has to be  $U(P)$  even or  $U(P)$  odd. In the degenerate case, we can have states with both parities. But, because of anti-periodic boundary conditions, any of them satisfies that  $U(P)\xi(\pi) = \xi(-\pi) = -\xi(\pi)$ . Thus, for any parity even state  $\xi_+$  this is possible only if  $\xi_+$  vanishes for  $\varphi = \pm\pi$ ,  $\xi_+(\pm\pi) = 0$ . In the same way since for any parity odd state  $\xi_-$  we have  $U(P)\xi_-(0_-) = \xi_-(0_+) = \xi_-(0)$ , any parity odd state vanishes for  $\varphi = 0$ , i.e.  $\xi_-(0) = 0$ . This property explains the vanishing of the heat kernel element  $K_T^{1/2}(0, \pi)$  for any value of  $T$ , previously derived by path integral methods, because *half* of the states of an orthonormal basis of stationary states in  $L^2(S^1)$  vanish at  $\varphi = 0$  whereas the other *half* vanish at  $\varphi = \pi$ .

Let us now consider the structure of the ground state  $\psi_0$ .

#### PROPOSITION 2.2

A planar rotor interacting with a transverse magnetic flux  $\Phi = \pi$  and a reflection invariant non-constant potential  $V(\varphi)$  with maximum height at  $\varphi = \pi$  and minimum value at  $\varphi = 0$  has a unique vacuum state  $\psi_0$  which is parity even and vanishes at  $\varphi = \pi$ .

*Proof.* Since the potential term  $V$  is non-trivial it gives a non-trivial contribution to the energy of stationary states. The states with lowest energy which are parity even vanish at  $\varphi = \pm\pi$ , where the potential terms attains its maximal value, and cannot have the same energy as parity odd states which vanish at  $\varphi = 0$ , where the potential terms attains its minimal value. This feature implies that the quantum vacuum state  $\psi_0$  is non degenerate, is parity even and vanishes at  $\varphi = \pm\pi$ . The splitting of energies between the ground state and the first excited state can also be understood in terms of tunnelling effect induced by instantons. But the argument used above is completely rigorous and does not rely on any semiclassical approximation or asymptotic expansion (see [4] for an early anticipation of this behaviour of the ground state based in numerical calculations).

The existence of a non-trivial potential with such a peculiar behaviour is crucial for the proof of the existence of vacuum nodes. If  $V = 0$ , there is no splitting between the energies of even and odd states and the ground state becomes degenerate. In this case there are ground states with indefinite parity which are linear combinations of parity even and

parity odd ground states and have no nodes. However, the kernel of the restriction of the operator  $K_T^{1/2}$  to the ground state subspace also vanishes for the pair of points 0 and  $\pi$  as the path integral formula predicts. The parity of the vacuum for general potentials with unique vacuum, depends on the structure of the potential. Although small generic perturbations of potentials of the type considered in Proposition 2.2 preserve the even character of the vacuum it might change for large perturbations due to the appearance of level crossings. The existence of such crossings for  $V = 0$  guarantees the consistence of the result when the maximum and minimum of  $V$  are interchanged.

The presence of a magnetic field generates nodes in the ground state as a vacuum response to the magnetic flux crossing the circle where the system evolves. This system mimics the behaviour of the 1+1 dimensional QED on a cylinder with a  $\theta$ -term  $\theta = 2\pi\epsilon$  when  $V = 0$ .

### 3 Hall Effect in a Torus

A charged quantum particle ( $e = m = 1$ ) moving on a two-dimensional torus  $T^2$  under the action of an uniform magnetic field  $B = k/2\pi$  ( $k \in \mathbb{Z}$ ) and an external potential  $V$  is governed by the Hamiltonian

$$H_A = -\frac{1}{2}\Delta_A + V, \quad (3.1)$$

where  $\Delta_A$  is the covariant Laplacian with respect to a  $U(1)$  gauge field  $A$  with curvature  $B$  defined on the line bundle  $E_k(T^2, \mathbb{C})$  with first Chern class  $k \in \mathbb{Z}$ , whose sections are the quantum states.

For trivial potentials  $V = \text{const}$ , the spectrum of the Hamiltonian (3.1) is exactly solvable. It is given by the Landau levels

$$E_n = B \left( n + \frac{1}{2} \right) \quad n \in \mathbb{N}$$

as in the infinite plane case. However, in the present case the degeneracy of each level is finite,  $\dim \mathcal{H}_0 = |k|$ , whereas in infinite volume the degeneracy is infinite for  $k \neq 0$ . The degeneracy of the ground state  $E_0$  is not only dependent of the first Chern class of the line bundle  $E_k(T^2, \mathbb{C})$ , where physical states are defined but also of the background metric of the torus and the form of the magnetic field [2].

#### 3.1 Translation anomalies

In the presence of the constant magnetic field,  $V = \text{const}$ , and a symmetric metric the classical system is translation invariant but the quantum generators of translation symmetries given by

$$\mathbf{L}^j = -iD_A^j - \epsilon^{jl}x_l B \quad j = 1, 2$$

suffer from an anomaly which transforms the abelian algebra  $\mathbb{R} \times \mathbb{R}$  into the Heisenberg algebra

$$[\mathbf{L}_1, \mathbf{L}_2] = -iB \quad (3.2)$$

as a central extension with central charge  $B$ . This is easy to understand because the system has two degrees of freedom and cannot have three independent commuting operators corresponding to time and space translations (3.1) (3.2).

In a  $T^2$  torus there is an extra anomaly of translation symmetry, for if the Heisenberg algebra were a real symmetry of the quantum system the energy levels would be infinitely degenerate, since any energy level supports a representation of the symmetry algebra and any representation of the Heisenberg algebra (3.1) must be infinite dimensional, but the energy levels do have a finite degeneracy  $k$ .

The presence of the anomaly is explicitly shown by the existence of a non-vanishing correlation function involving the time derivative of the would be conserved currents  $l_j = \dot{x}_j - \epsilon_{jn} x^n B$  associated to translation transformations

$$\langle \psi_0 | \dot{l}_j(t) \dot{x}_n(s) | \psi_0 \rangle = -i\theta_3(0)(\delta_{jn} + i\epsilon_{jn}) \frac{B^2}{2} e^{iB(s-t)},$$

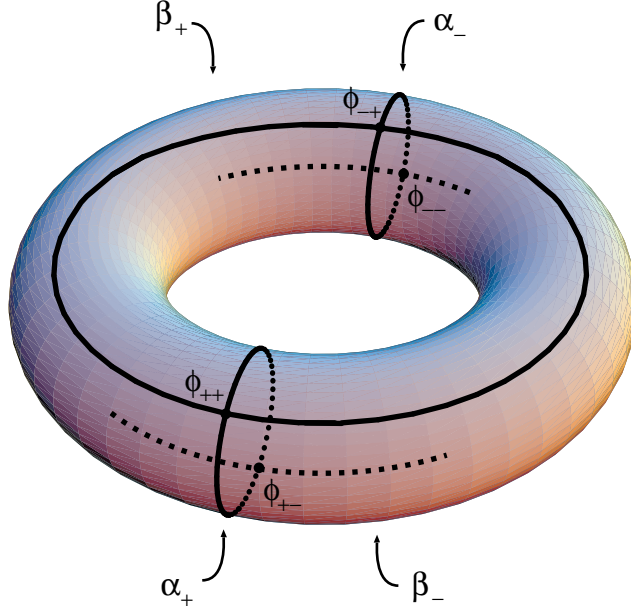
where

$$\theta_3(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/2} e^{2nui}$$

is the third Jacobi theta function. This is the simplest example of an anomalous symmetry in a quantum mechanical system. Notice that it is not present in the infinite volume limit.

There is an operator theory explanation for this anomaly [3] [1]. Although the generators of the translation Heisenberg algebra  $\mathbf{L}_1, \mathbf{L}_2$  commute with  $\mathbf{H}$  on the domain of functions with compact support on  $(0, 2\pi) \times (0, 2\pi)$ , the corresponding selfadjoint extensions *do not* commute, because the domain of definition of  $\mathbf{H}$  is not preserved by the action of  $\mathbf{L}_j$ . In this sense translation invariance is broken in the quantum system. This interpretation of the anomaly based on the anomalous behaviour of the domain of definition of the quantum Hamiltonian under translations was first pointed out by Esteve [8] and Manton [9]. In this case the existence of an anomalous commutator is crucial for the understanding of the finite degeneracy of energy levels in spite of the existence of a partial translation invariance.

There is a simpler geometrical interpretation of the anomaly. The quantum system is not completely specified by the magnetic flux. To define the connection  $A$  one has to specify the holonomies,  $h_\alpha(A)$ ,  $h_\beta(A)$ , along two complementary non-contractible circles of the torus  $\alpha, \beta$ . Once  $h_\alpha(A)$ ,  $h_\beta(A)$  are specified, the holonomy along any other closed loop on  $T^2$  is completely determined because the holonomies along two homotopically equivalent circles differ by a phase factor whose exponent is twice the magnetic flux of the torus domain enclosed by them. This means that while any of the basic circles  $\alpha, \beta$  sweeps the torus under a  $2\pi$  translation its holonomy describes a non-contractible loop along the gauge group  $U(1)$ . Thus, there are at least two non-contractible circles  $\alpha_0, \beta_0$  on the torus at which the holonomies of  $A$  reduce to the identity, i.e.  $h_{\alpha_+}(A) = h_{\beta_+}(A) = I$ . In a similar way there are at least other two non-contractible circles  $\alpha_-, \beta_-$  at which the the holonomies of  $A$  are minus the identity, i.e.  $h_{\alpha_-}(A) = h_{\beta_-}(A) = -I$  (see fig. 1).



**Figure 1.** Circles with holonomies  $\pm 1$  and nodal points of quantum vacua

These loops are unique for any connection with unit first Chern class, i.e.  $c(A) = k = 1$ . The existence of such special loops explains why translation symmetry is completely broken for  $k = 1$  in the Hall effect on a torus. Only translations which give a complete turn to the torus leave the Hamiltonian invariant. The translation symmetry group is then reduced from  $T^2$  to  $I$ . For the same reasons for higher values of  $k$  the number of closed circles with trivial holonomy in each homotopy class is equal to  $k$ . If  $k > 1$  there are  $k$  circles in the same homotopy class with the same holonomy. This means that the continuous translation symmetry is reduced by the anomalies to a discrete quantum symmetry generated by translations by an angle  $2\pi/k$  in each of the two transversal directions of the torus, i.e. the symmetry is reduced from  $T^2$  to a central extension of  $\mathbb{Z}_{k-1} \times \mathbb{Z}_{k-1}$ .

### 3.2 Parity anomaly

There is another discrete symmetry which also becomes anomalous upon quantization. Let us introduce angular coordinates on the torus,  $T^2 = [0, 2\pi) \times [0, 2\pi)$ . The classical system is invariant under the combined action of two reflections with respect to any pair of angles  $\phi = (\phi_1, \phi_2)$ ,

$$P_{\phi_1}(\varphi_1, \varphi_2) = (2\phi_1 - \varphi_1, \varphi_2) \quad P_{\phi_2}(\varphi_1, \varphi_2) = (\varphi_1, 2\phi_2 - \varphi_2)$$

i.e.  $P_\phi = P_{\phi_1} P_{\phi_2}$ . Notice that any of these parity transformations  $P_{\phi_1}, P_{\phi_2}$  is not a symmetry because it reverses the orientation of the torus and, thus, the sign of the magnetic field.

However, the need of specification in the quantum theory of the holonomies  $h_\alpha(A), h_\beta(A)$  breaks down this reflection symmetry with respect to the a generic point of the torus  $(\phi_1, \phi_2)$  except for the four crossing points  $\phi_{++} = (\phi_1^+, \phi_2^+)$ ,  $\phi_{+-} = (\phi_1^+, \phi_2^-)$ ,  $\phi_{-+} = (\phi_1^-, \phi_2^+)$ ,



$\phi_{--} = (\phi_1^-, \phi_2^-)$  of the circles with holonomies  $I$  or  $-I$  (see Fig. 1),

$$\alpha_{\pm}(\varphi) = (\phi_1^{\pm}, \varphi) \quad \beta_{\pm}(\varphi) = (\varphi, \phi_2^{\pm}) \quad \varphi \in [0, 2\pi).$$

Reflection with respect to any other point transforms loops into loops with different holonomy for  $k = 1$ . For  $k > 1$  there are more crossing points of circles with holonomies  $I$  and  $-I$ , thus, the reflection symmetry group is bigger in that case. In any case the remaining quantum symmetry,  $U(P_{\pm\pm})$  defined by

$$U(P_{\pm\pm})\psi(\varphi_1, \varphi_2) = e^{-i(\phi_2^{\pm}\varphi_1 - \phi_1^{\pm}\varphi_2)}\psi(U(P_{\pm\pm})(\varphi_1, \varphi_2)) \quad (3.3)$$

is very relevant to find the nodes of the ground states.

### 3.3 Vacuum Structure

Since the line bundle  $E_k(T^2, \mathbb{C})$  is non-trivial for  $k \neq 0$  any section must have nodal points. This means that any energy level has a non-trivial nodal structure. For  $|k| > 1$  the degeneracy of energy levels is  $k$  which means that given any point  $\phi_*$  on the torus there is one state in that level with a node at  $\phi_*$ . Therefore in such a case the physical meaning of the nodal configuration cannot be relevant. However, in the case  $k = 1$  ( $B = 1/2\pi$ ) there is no degeneracy in the energy levels and the vacuum do have only one node which certainly has to be a very distinguished classical configuration for the quantum system. The search of vacuum nodes is simplified for the case of  $P_{++}$ -symmetric potentials  $V(P_{++}\varphi) = V(\varphi)$ , by the following result.

#### PROPOSITION 3.1

For any  $P_{++}$  symmetric potential  $V$  the heat kernel elements

$$K_T^A(\phi_{++}, \phi_{--}) = K_T^A(\phi_{+-}, \phi_{--}) = K_T^A(\phi_{-+}, \phi_{--}) = 0 \quad (3.4)$$

vanish for any  $T$  if  $k = 1$ .

*Proof.* The basic strategy is similar to that used in the planar rotor. Because of the non-degeneracy of the energy levels, any stationary state must have a definite  $U(P_{\pm\pm})$ -parity symmetry with respect to the four points  $\phi_{++}, \phi_{+-}, \phi_{-+}, \phi_{--}$  where the circles with holonomies  $I$  and  $-I$  cross each other. There are four quantum parity symmetry transformations  $U(P_{++})$ ,  $U(P_{+-})$ ,  $U(P_{-+})$  and  $U(P_{--})$ . Although, the four transformation are identical in  $T^2$ , e.g. all of them leave the four points invariant, they define four different unitary transformations in the space of quantum states. If we redefine our coordinates so that  $\phi_1^+ = \phi_2^+ = 0$ , we have that  $\phi_1^- = \phi_2^- = \pi$ . In such coordinates  $\varphi = (\varphi_1, \varphi_2)$  the gauge field with the required holonomy properties is given by

$$A_i = \frac{B}{2}\epsilon_{ij}\varphi^j \quad (3.5)$$

in a gauge with boundary conditions

$$\psi(\varphi_1 + 2\pi, \varphi_2) = e^{i\pi B \varphi_2} \psi(\varphi_1, \varphi_2) \quad \psi(\varphi_1, \varphi_2 + 2\pi) = e^{-i\pi B \varphi_1} \psi(\varphi_1, \varphi_2). \quad (3.6)$$

Since the  $P_{++}$  symmetry leaves the point  $\phi_{--}$  invariant, physical states  $\psi$  must verify  $U(P_{++})\psi(\pi, \pi) = \psi(-\pi, -\pi) = e^{i\pi} \psi(\pi, \pi) = -\psi(\pi, \pi)$ . Thus, if  $\psi$  is  $U(P_{++})$  even  $\psi$  has a node at  $\phi_{--}$ , i.e.  $\psi(\phi_{--}) = 0$ . For the same reason

$$U(P_{++})\psi(0, 0) = \psi(0, 0), \quad U(P_{++})\psi(0, \pi) = \psi(0, -\pi) = \psi(0, \pi),$$

$$U(P_{++})\psi(\pi, 0) = \psi(-\pi, 0) = \psi(\pi, 0),$$

which implies that  $P_{++}$  odd states must vanish at  $\phi_{++}$ ,  $\phi_{-+}$  and  $\phi_{+-}$ , i.e.  $\psi(\phi_{++}) = \psi(\phi_{+-}) = \psi(\phi_{-+}) = 0$ . In a similar way we get that

$$U(P_{--})\psi(\pi, \pi) = \psi(\pi, \pi), \quad U(P_{--})\psi(0, \pi) = e^{i\pi/2} \psi(2\pi, \pi) = -\psi(0, \pi),$$

$$U(P_{--})\psi(0, 0) = \psi(2\pi, 2\pi) = e^{-i\pi} \psi(0, 2\pi) = -\psi(0, 0),$$

$$U(P_{--})\psi(\pi, 0) = e^{-i\pi/2} \psi(\pi, 2\pi) = -\psi(\pi, 0),$$

which implies that  $P_{--}$  odd states must vanish at  $\phi_{--}$ , whereas  $P_{--}$  even states must vanish at  $\phi_{++}$ ,  $\phi_{-+}$  and  $\phi_{+-}$ . Similar properties hold for the remaining parity operators  $U(P_{-+}), U(P_{+-})$ . Since there is a complete basis of stationary states consisting of  $U(P_{--})$  even and  $U(P_{--})$  odd states, this implies the vanishing of the kernel matrix elements (3.4).

As in the planar rotor case there is an alternative derivation of the same results. It is based on the path integral approach. The method also carries enough information to identify the parity of the vacuum state. The essential feature is to prove that the matrix element  $K_T^A(\phi_{++}, \phi_{--}) = \langle \phi_{++} | e^{-TH_A} | \phi_{--} \rangle$  of the euclidean time evolution kernel vanishes for any  $T$ . This property can be easily derived from the path integral representation of the heat kernel

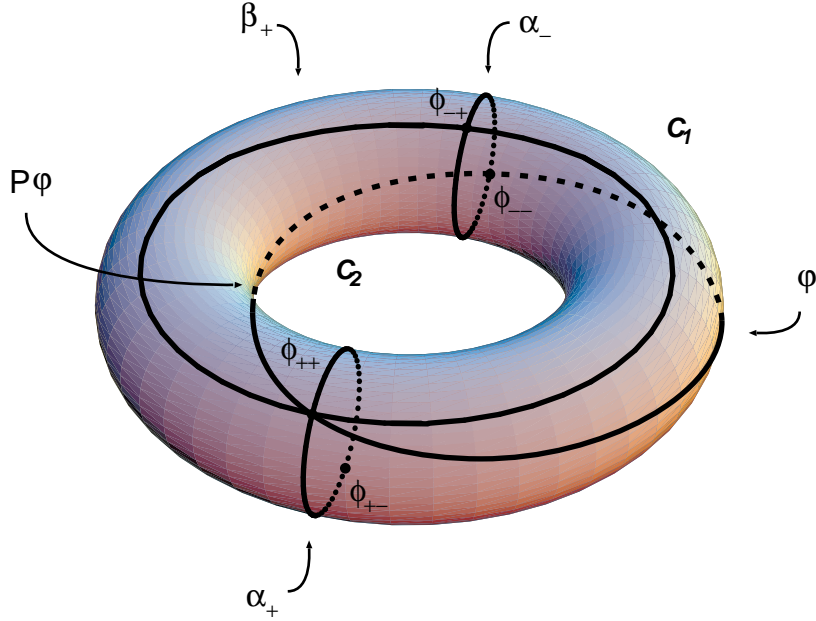
$$K_T^A(\phi_{++}, \phi_{--}) = \int_{\substack{\varphi(0)=\phi_{++} \\ \varphi(T)=\phi_{--}}} \delta\varphi \quad h_\varphi^A(t) \quad \exp \left\{ - \int_0^T dt \left[ \frac{1}{2} \dot{\varphi}(t)^2 + V(\varphi(t)) \right] \right\}. \quad (3.7)$$

where  $h_\varphi^A(t)$  is the holonomy of the closed path  $\varphi(t)$ . In the path integral representation (3.7) a path  $\varphi(t)$  connecting  $\phi_{++}$  and  $\phi_{--}$  transforms under the  $P$  reflection symmetry into another path  $P\varphi(t)$  which connects the same points and gives the same contribution to the real term of the exponent in the path integral. However, the contribution of both paths to the imaginary part is different. They contribute to the path integral with a phase factor which is exactly the holonomy of  $A$  along the paths. It is immediate to see that the ratio of both contributions  $h_\varphi^A(t)(h_{P\varphi(t)}^A)^{-1}$  equals the holonomy of the closed loop obtained by composition  $\varphi(t) \circ P\varphi(T-t)$  which is in the homotopy class  $(2n_1 + 1, 2n_2 + 1)$  of  $\alpha_-^{2n_1+1} \circ \beta_+^{2n_2+1}$ . The holonomy splits, by Stokes theorem, into a factor which is the holonomy of the basic circles

$\alpha_-^{2n_1+1} \circ \beta_+^{2n_2+1}$  and the magnetic flux  $\Phi_1, \Phi_2$  crossing two surfaces  $\mathcal{C}_1$  and  $\mathcal{C}_2$  in  $T^2$ :  $\mathcal{C}_1$  being the domain of  $T^2$  enclosed by the curves  $\varphi(t)$  and

$$\gamma_1(t) = \begin{cases} ((2n_1 + 1)\pi + (2n_1 + 1), 0) \frac{2\pi}{T}(t - \frac{T}{2}) & 0 \leq t \leq \frac{T}{2} \\ ((2n_1 + 1)\pi, (2n_2 + 1)\pi + (2n_2 + 1) \frac{2\pi}{T}(t - T)) & \frac{T}{2} \leq t \leq T, \end{cases} \quad (3.8)$$

and  $\mathcal{C}_2$  being the surface enclosed by  $P\varphi(T - t)$  and  $P\gamma_1(T - t)$  (see Fig. 2). Those contributions of magnetic fluxes are opposite and cancel each other. Thus, the contribution of  $h_\varphi^A(t)(h_{P\varphi(t)}^A)^{-1}$  is reduced to the holonomy of  $\alpha_-^{2n_1+1} \circ \beta_+^{2n_2+1}$ , i.e.  $(-1)^{2n_1+1} = -1$ . This means that the contributions of  $\varphi(t)$  and  $P\varphi(t)$  to the path integral are equal but with opposite signs. The contributions of both paths cancel and the argument can be repeated path by path to show that the whole path integral vanishes. In a similar way we can prove the vanishing of the kernel element  $K_T^A(\phi_{+-}, \phi_{--}) = K_T^A(\phi_{-+}, \phi_{--}) = 0$  for  $\phi_{+-}$  and  $\phi_{-+}$ , because in that case the corresponding holonomies of paths and reflected paths differ by the holonomies of the loops  $\alpha_-^{2n_1+1} \circ \beta_+^{2n_2}$  and  $\alpha_+^{2n_1} \circ \beta_-^{2n_2+1}$ , respectively. The relative negative sign is again the basis for the cancellation of the corresponding contributions to the path integral. Notice, however, that the argument cannot prove the vanishing of  $K_T^A(\phi_{-+}, \phi_{++})$ ,  $K_T^A(\phi_{+-}, \phi_{++})$  or  $K_T^A(\phi_{+-}, \phi_{-+})$ .



**Figure 2.** Paths giving opposite contributions to the path integral kernel

Let  $P_{\uparrow\uparrow}$  denote the reflection operator with respect to the crossing points  $\phi_{\uparrow\uparrow}$  where the two circles with holonomy  $iI$  cross each other for  $k = 1$ .

### THEOREM 3.1

Let  $k = 1$  and  $V$  be an invariant potential under reflections  $P_{++}$  and  $P_{\uparrow\uparrow}$ . The ground state

is unique,  $U(P_{++})$  even and has a node at  $\phi_{++}$ .

*Proof.* In the case  $V = 0$  the parity behaviour of the vacuum state can be obtained from an indirect argument. We know that the ground states in absence of potential term are holomorphic sections of the line bundle  $E_k(T^2, \mathbb{C})$  with Chern class  $c(E_k) = k$  (see Ref. [2] for a review and references therein). Any holomorphic section of  $E_k$  can have only  $k$  single nodes. For  $k = 1$  only  $P_{++}$ -even states can have only one single node at  $\phi_{--}$ , whereas  $P_{++}$ -odd states have at least three-different nodal points. Thus, the vacuum state is  $P_{++}$ -even and has a node at the crossing of the two circles with holonomy  $-I$  (the corresponding Abrikosov lattice has one single vortex). The same state is parity odd with respect to  $P_{--}$ ,  $P_{+-}$  and  $P_{-+}$ . (The property also holds for  $k > 1$ ). Those results can be explicitly checked from the exact analytic solutions

$$\begin{aligned} \psi_{nl}(\varphi_1, \varphi_2) &= \frac{1}{2\pi} \left(\frac{2}{k}\right)^{\frac{1}{4}} \frac{1}{\sqrt{n!2^n}} e^{i\frac{k}{4\pi}\varphi_1\varphi_2} \sum_{m \in \mathbb{Z}} e^{im(\varphi_1 + 2\pi\frac{l}{k})} \\ &\times H_n \left( \sqrt{\frac{k}{2\pi}} \left( \varphi_2 + 2\pi\frac{m}{k} \right) \right) e^{-\frac{1}{4\pi k}(2\pi m + k\varphi_2)^2} \\ l &= 0, \dots, k-1 \end{aligned} \quad (3.9)$$

However, the symmetry arguments already introduced in the case of the planar rotor allows us to generalize this result for more general potentials which makes it extremely useful especially for non-exactly solvable cases.

The vanishing of  $K_T^A(\phi_{++}, \phi_{--}) = 0$  for any  $T$  implies that

$$\sum_n U(P_{++})\psi_n(\phi_{++})^* U(P_{++})\psi_n(\phi_{--}) = 0$$

for any energy level.

If the ground state is degenerate there are at least two states  $\psi_0^+$  and  $\psi_0^-$  which are even and odd, respectively, with respect to  $P_{++}$ -parity.  $\psi_0^+$  vanishes at  $\phi_{--}$  and  $\psi_0^-$  at  $\phi_{-+}$ ,  $\phi_{+-}$  and  $\phi_{++}$ . We know that the kinetic term contribution is minimized in a state with a single node at  $\phi_{--}$ . Since the potential term is reflection symmetric it has the same behaviour near the four points. Thus, the kinetic and potential energies are minimized on states with a unique node at  $\phi_{--}$  instead of three nodes at  $\phi_{-+}$ ,  $\phi_{+-}$  and  $\phi_{++}$ . From Ritz's variational argument it is obvious that  $\psi_0^+$  and  $\psi_0^-$  cannot have the same energy. The existence of such non-trivial splitting implies that the vacuum state  $\psi_0$  is unique and, thus, has a unique node at  $\phi_{--}$  and is even with respect to  $P_{++}$ -parity and odd with respect to  $P_{-+}$ ,  $P_{+-}$  and  $P_{--}$  parities.

It is also easy to understand this result from perturbation theory, because  $V$  does not connect parity even states  $\psi_+$  with parity odd states  $\psi_-$ . In fact, by parity symmetry  $\langle \psi_+ | V^n | \psi_- \rangle = 0$ , which implies that there are no corrections to the parity behaviour of the ground state at any order in perturbation theory. This result is, thus, compatible with our non-perturbative result which holds for larger potentials and tells us that there is no

level crossing of ground states whenever we keep the reflection symmetry properties of the potentials.

We have found that the vacuum state vanishes at the intersection of the only pair of circles with holonomy  $-I$ . The singularity of this point explains very explicitly why translation invariance is completely broken. Only translations by  $2\pi$  can leave the quantum states and their nodes invariant. For higher values of  $k$  we get similar results but now there are  $k$  circles with holonomy  $-I$  in each direction which cross at  $k^2$  different points. Any of such points is one of the  $k$  nodes of parity even states under reflections with respect to the opposite crossing point of two circles with unit holonomy. Those nodes give rise to the Abrikosov lattice of vortices in a type II planar superconductor. But now we have different states vanishing at the different intersections which transform one into each other by the remaining discrete symmetries. In addition there are linear combinations of parity even and odd states for each level which vanish elsewhere on  $T^2$ . Therefore, nothing special happens at that intersections. Although in the  $k = 1$  case the points  $\phi_{++}$ ,  $\phi_{--}$ ,  $\phi_{-+}$  and  $\phi_{+-}$  are distinguished points because they are the only nodes of stationary states, they do not have any physical meaning because the breaking of translation symmetry is reflected in the fact that there is a  $T^2$  moduli space of  $U(1)$ -connections  $A$  which generate the same magnetic field but differ by their holonomies, and for any point  $\phi$  of  $T^2$  there is a connection whose ground state vanishes at  $\phi$ . All these connections and their corresponding eigenstates are obtained from those of one fixed connection by translations. Therefore the location of the vacuum nodes in this case does not have an special meaning. This is in contrast with what happens in quantum field theories, where the quantum vacua are really unique and their nodes are very special field configurations carrying, therefore, a relevant dynamical information [5]–[7].

The same arguments also apply for higher genus surfaces or surfaces with holes, giving very relevant information on the structure of the Abrikosov lattice for systems with defects.

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